Implicit Surfaces - Lab 4 TNM079

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Abstract

The paper discusses Implicit surfaces and the reasons why they are effective and popular in computer graphics. It focuses on a simple implementation of implicit surfaces through a modeling framework called constructive solid geometry (CSG) where operations such as union, difference or intersection are included. This lab will only process analytical surfaces such as planes, cuboid or spheres.

1 Introduction

Implicit surfaces work well in sumulations since they are assured to be physically realistic. They also don't contain any holes or self-intersections. An implicit surface is a surface in Euclidian space defined by an equation. Since all analytical surfaces can be described through different weights and combination of x, y, z can an abritrary analytical surface be described by a $4x4$ matrix.

2 Background

Implicit functions is an indirect representation of surfaces since the surface are defined by an equation which is to be solved to find or approximate the geometry.

$$x = \pm\sqrt{R^2 - y^2}. \tag{1}$$

$$f(x, y) = x^2 + y^2 \tag{2}$$

Equation 1 is an explicit function that describes a circle while 2 defines it implicitly. In 2, the function f designates a scalar value to all points in the xy-plane. The scalar value is called an *isovalue*, C, which generates a subset of the xy-domain with that is called *level-set* or *isosurface*. Isovalue $C = R^2$ finds the level-set that describes the circle in equation 2. The level-set of a scalar function will have one lower dimension than its scalar function, so a 2D scalar function turns into a 1D contour line. The difference between explicit and implicit surfaces might not be clear in the example shown in equation 1 and 2 but in more complicated cases is the implicit representation vastly more straightforward.

An advantage of implicit representations is that it is easy to define points that are either inside, outside or on the surface. Given an isovalue $C = f(\mathbf{x})$, we can classify point $\mathbf{x}$ as follows:

$$\begin{array}{ll} \text{Inside:} & f(\mathbf{x}) < C \\ \text{Outside:} & f(\mathbf{x}) > C \\ \text{On surface:} & f(\mathbf{x}) = C \end{array} \tag{3}$$

This is a property that is used in the union or intersection operations. Another strength of implicit surfaces comes with efficiency of computing differential attributes. A good example for this is the surface normals. The expression of the normal can be derived of a 3D scalar field by looking at the gradient ∇ of the scalar field $f(x, y, z)$. If a coordinate frame can be found where one of its basis vectors shares the direction of the normal can the normal gradient function be rewritten to

$$\begin{aligned} \nabla = \quad & \vec{e}_1 \tfrac{\partial}{\partial \vec{e}_1} + \vec{e}_2 \tfrac{\partial}{\partial \vec{e}_2} + \vec{n} \tfrac{\partial}{\partial \vec{n}} = \\ & \vec{e}_1 \left(\vec{e}_1 \cdot \nabla\right) + \vec{e}_2 \left(\vec{e}_2 \cdot \nabla\right) + \vec{n} \left(\vec{n} \cdot \nabla\right) \end{aligned} \tag{4}$$

This gives:

$$\nabla f = \vec{e}_1 \left(\vec{e}_1 \cdot \nabla f\right) + \vec{e}_2 \left(\vec{e}_2 \cdot \nabla f\right) + \vec{n} \left(\vec{n} \cdot \nabla f\right) \tag{5}$$

A manifold level-set will contain infinitesimally small patches of the surface that will be flat. This means that if one vector in the coordinate frame has the same direction as the normal will the other two vectors lie in the tangent plane and thus on the surface itself. The scalar field f will then be locally constant everywhere which results in

$$\begin{aligned} \tfrac{\partial f}{\partial \vec{e}_1} = \tfrac{\partial f}{\partial \vec{e}_2} \quad &= \quad 0 \\ &\Rightarrow \quad \nabla f = \vec{n}(\vec{n} \cdot \nabla f) \\ &\Rightarrow \quad \vec{n} = \tfrac{\nabla f}{|\nabla f|} \end{aligned} \tag{6}$$

if we use the sign convention in equation 3. To calculate the differential of an implicit surface is a discrete scheme required. There's multiple different scheme with various benefits, two of which is either forward-differential, 7, or central diffence approximation 8.

$$D_x \equiv \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \approx \frac{f(x_0 + \epsilon) - f(x_0)}{\epsilon} \tag{7}$$

$$\begin{aligned} D_x \equiv \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \approx \\ \approx \frac{f(x_0 + \epsilon) - f(x_0 - \epsilon)}{2\epsilon} \end{aligned} \tag{8}$$

For the second derivative can equation 9 be used.

$$\frac{\partial^2 f}{\partial x^2} = D_{xx} \approx \frac{f(x_0 + \epsilon) - 2f(x_0) + f(x_0 - \epsilon)}{\epsilon^2} \tag{9}$$

As explained previously is the implicit representation be defined as a function $f(x, y, z)$ and thus can a quadric function be described as:

$$\begin{aligned} f(x, y, z) \quad &= \quad Ax^2 + 2Bxy + 2Cxz \\ &+ \quad 2Dx + Ey^2 + 2Fyz \\ &+ \quad 2Gy + Hz^2 + 2Iz \\ &+ \quad J \end{aligned} \tag{10}$$

or on matrix form:

$$\mathbf{p}^T \mathbf{Q} \mathbf{p} = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{11}$$

Since a quadric surface is known analytically can the differentiation be applied to the quadric directly rather than using equation 6. This gives an efficient and analytical expression with a coefficient matrix Q. Using the matrix form can numerous analytical geometries be formed through implicit surfaces in a very effective way.

$$\begin{aligned} \nabla f(x, y, z) = 2 \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \\ = 2\mathbf{Q}_{sub}\mathbf{p} \end{aligned}$$

3 Results

The implementations of the implicit functions were very straightforward and quick to implement. The first task was to complement the existing `CSG.h` to include all operators such as union and intersection. This was done by implementing boolean operators on the isovalue of the two incoming implicit surfaces. If the minimum of the two was returned we got `union`, if the maximum of the two was returned `intersection`. A `difference` operator was also implemented that returned the maximum of implicit surfaces A_1 and $-A_2$.

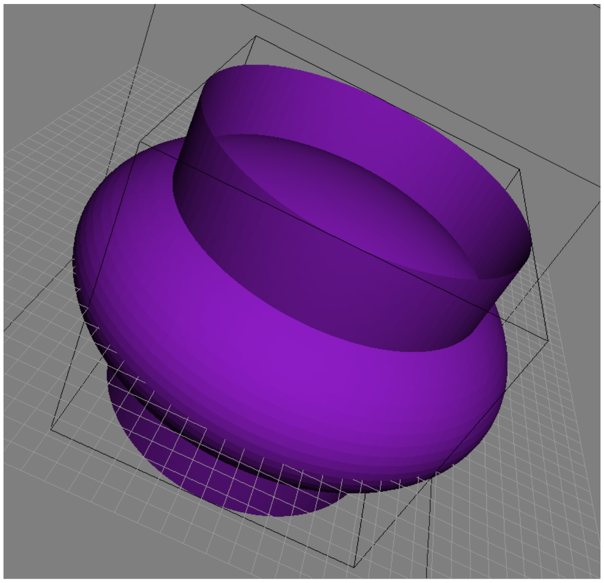

Figure 1: Two implicit surfaces, an ellipsoid and a cylinder

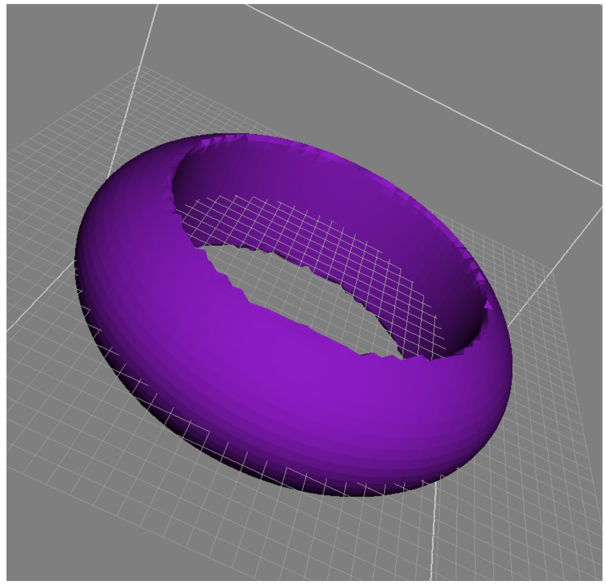

Figure 2: Showing the difference operator between two implicit surfaces

The next task was to implement the quadric implicit surfaces where equations 11 and 12 were used. Various different basic primitives, such as sphere, cuboid or ellipsoid, with the quadric implicit functions were also defined in `FrameMain`.

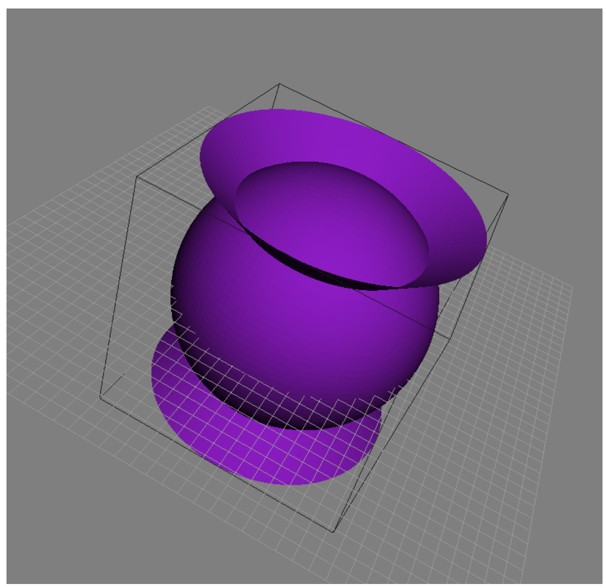

Figure 3: Two implicit surfaces, a sphere and a cylinder

References

[1] M. E. Dieckmann, *Lecture Slides for TNM079*, **Lecture 6**, 2016.

This report aims for grade 3.

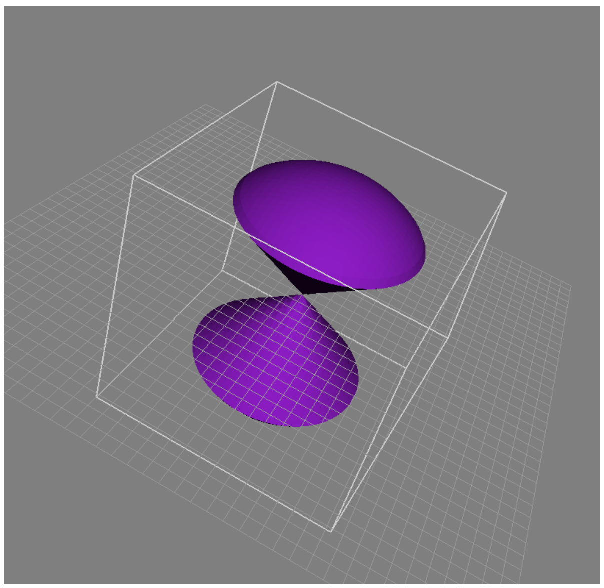

Figure 4: Showing the intersection operator between two implicit surfaces