Level-sets - Lab 5 TNM079

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Abstract

The paper discusses the theory and implementation of a level set framework. A level set is a subset of an implicit surface which can be deformed by solving a number of *partial differential equations* (PDEs). The implicit representation comes with the benefit of easy changes in topology and formation. Stable schemes on both *hyperbolic* and *parabolic* differentials are discussed and will allow for erosion, dilation and advection.

1 Introduction

Level sets are a subset of an implicit function and are commonly used in conjunction with deformation operations such as erosion, dilation or advection. Since the implicit representation makes it easy to define distance from the surface is this an efficient approach. The method is defined as an interface as a level set S of the level set function ϕ as

$$S = \{\vec{x} \in \Re^d : \phi(\vec{x}) = h\} \tag{1}$$

where points are inside S when $\phi(vecx) < h$ and outside when $\phi(vecx) > h$.

2 Background

Furthermore, the normal of any level set of ϕ can be defined as:

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} \tag{2}$$

To be able to manipulate the topology are the equations of motion for the level set derived. This is done by introducing time dependency to equation 1. There are multiple ways of achieving this, one of which is called *static level set formulation* and allows the isovalue vary over time as $h(t)$. It will describe how the level set of a function evolves as the isovalue changes. The downfall of this method is that the level set cannot intersect by its definition, limiting the possible deformations. Another option is to change the level set function itself over time, such that $S(t) = \{\vec{x} \in \Re^d : \phi(\vec{x}, t) = h\}$. $\alpha(t)$ is observed in order to derive the equations of motion for S. Since $\alpha(t)$ is on and follows the movement of S, it is known that $\phi(\alpha(t), t) \equiv h$. The time differentation becomes:

$$\frac{\partial\phi}{\partial t} = -\nabla\phi \cdot \frac{d\boldsymbol{\alpha}}{dt} \tag{3a}$$

$$= -F\,|\nabla\phi| \tag{3b}$$

where F is referred to as the *level set speed function*:

$$F = \boldsymbol{n} \cdot \frac{d\boldsymbol{\alpha}}{dt} = \frac{\nabla\phi}{|\nabla\phi|} \cdot \frac{d\boldsymbol{\alpha}}{dt} \tag{4}$$

This provides the user with all the means of manipulating the level set function to achieve a desired motion on the surface S. The choice of h in the equation is arbitrary. However, using $h = 0$ has benefits as it allows the definition of *inside* and *outside* the surface to be done through simple sign convention. Equation 3 is a continuous derivation on the level set function, but to apply the theory in computer graphics it is necessary to discretize both the temporal-and spatial domain. This can have a number of numerical implications, depending on what PDE that is used.

First off, the temporal discretization decides how equation 3 will change over discrete timesteps Δt. Implicit schemes are stable regardless of timestep, however they can be computationally heavy. Despite the constraints explicit methods introduce they are often used to solve level sets due to their efficiency. A simple explicit scheme is the *forward Euler*:

$$\frac{\partial \phi}{\partial t} \approx \frac{\phi^{n+1} - \phi^n}{\Delta t} \tag{5}$$

where ϕ^n defines the values of ϕ at time instance t^n while ϕ^{n+1} at the time instance $t^n + \Delta t$. For better accuracy could the Euler scheme be changed to a *total variation diminishing Runge-Kutta* method at the cost more computations.

The spatial discretization relies strongly on what PDE is used and thus will two fundamental types be used, *hyperbolic* and *parabolic*. For hyperbolic advection two versions of equation 3 are used:

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= -\boldsymbol{V} \cdot \nabla \phi && \text{(6a)} \\ &= -F\,|\nabla \phi| && \text{(6b)} \end{aligned}$$

Equation 6a describes advecting the interface in a vector field $\boldsymbol{V}$ and equation 6b the direction of the surface normal. An equation similar to this can be used in erosion and dilation. Since the sample points in front of current position have not been ẗouchedït can be deduced that only sample points behind (or *up-wind* to) the current position should be used in the discretization. This results ina finite difference approximation:

$$\frac{\partial \phi}{\partial x} \approx \begin{cases} \phi_x^+ = \left(\phi_{i+1,j,k} - \phi_{i,j,k}\right)/\Delta x & \text{if } V_x < 0 \\ \phi_x^- = \left(\phi_{i,j,k} - \phi_{i-1,j,k}\right)/\Delta x & \text{if } V_x > 0 \end{cases} \tag{7}$$

Equation 6b does not know the direction of the flow, to deduce this can Godunov's method to evaluate partial derivatives be used. This result in a first order accurate approximation:

$$\left(\frac{\partial \phi}{\partial x}\right)^2 \approx \begin{cases} \max\left[\max(\phi_x^-, 0)^2, \min(\phi_x^+, 0)^2\right] & F > 0 \\ \max\left[\min(\phi_x^-, 0)^2, \max(\phi_x^+, 0)^2\right] & F < 0 \end{cases} \tag{8}$$

Parabolic type is often used to smooth deformations and can be stated as:

$$\frac{\partial \phi}{\partial t} = \alpha\kappa\,|\nabla \phi| \tag{9}$$

where α is a scaling parameter and κ is curvature. A big difference with this equation to the hyperbolic representation is that it has no direction. A second-order accurate central difference scheme for space is required:

$$\frac{\partial \phi}{\partial x} \approx \phi_x^{\pm} = \frac{\phi_{i+1,j,k} - \phi_{i-1,j,k}}{2\Delta x} \tag{10}$$

And second order central difference scheme:

$$\frac{\partial^2 \phi}{\partial^2 x} \approx \frac{\phi_{i+1,j,k} - 2\phi_{i,j,k} + \phi_{i-1,j,k}}{\Delta x^2} \tag{11a}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} \approx \frac{\phi_{i+1,j+1,k} - \phi_{i+1,j-1,k} + \phi_{i-1,j-1,k} - \phi_{i-1,j+1,k}}{4\Delta x \Delta y} \tag{11b}$$

The stability constraints required in the hyperbolic type for the time step Δt is not necessary here as information travel at inifinite speed

As previously mentioned, ϕ needs to be continuous with well defined gratidents in order for use of equation 3. However with the discretization this constraint can be relaxed. The gradients of ϕ /textitcan be discontinuous, although the *rate-of-change* of ϕ must be bounded by the finite Lipschitz constant $Cgeq0$. To assure stability, $C \approx 1$, meaning that ϕ has to satisfy the Eikonal equation:

$$|\nabla \phi| = 1 \tag{12}$$

To ensure that $|\nabla \phi|$ does not drift away from 1, a process called *reinitialization* is called frequently which ensures a steady state by

$$\frac{\partial \phi}{\partial t} = S(\phi)(1 - |\nabla \phi|) \tag{13}$$

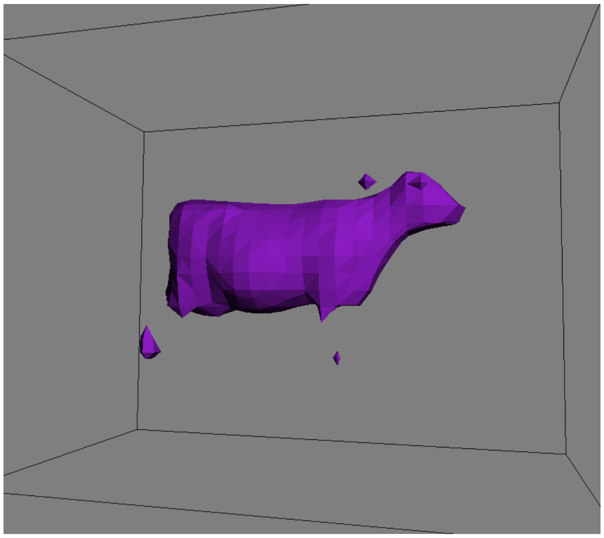

Figure 1: A broken model with submeshes around the main geometry

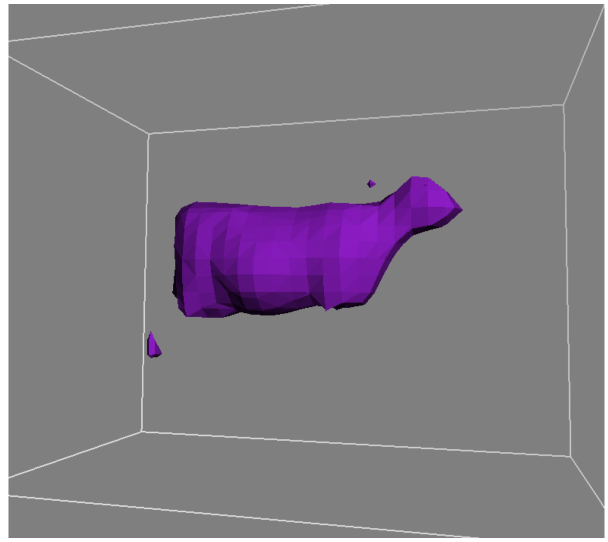

Figure 2: Dilating to close the gaps between the submodels

where $S(\phi)$ is the *sign* of ϕ, with a smooth numerical approximation.

3 Results

The first task was to implement the differential equations with both a forward, backward and central difference scheme using the equations given in 7 and 10. The second derivative scheme was also implemented using equation 11. This was done with respect to all three axises.

These equations were later used when evaluating the advection a vector field has on the mesh. If the vector field stepped backwards in the $x - axis$, it means that the backward-scheme will be used to calculate the gradients first value. By evaluating each variable of the vector field individually an appropriate differing scheme can be selected.

If a deformation of the mesh along the surface normal is desired will equation 6b be used. To evaluate the sign of the second derivative was the Godunov scheme used.

Figure 1, 2 and 3 shows how small submeshes of the model can be removed by first using dilation and the erosion. The technique is often called *closing*.

References

[1] M. E. Dieckmann, *Lecture Slides for TNM079*, **Lecture 7**, 2016.

This report aims for grade 3.

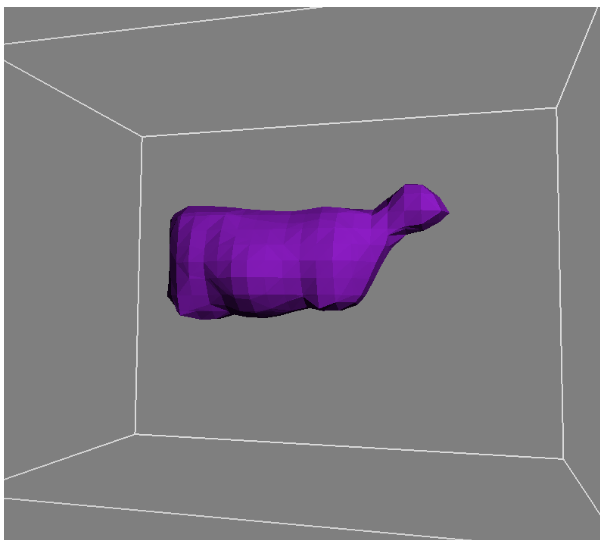

Figure 3: Call on erode to smooth out the surface, rendering the result without the previous floating spheres around the model.